ELSEVIER
Journal of Computational and Applied Mathematics 50 (1994) 401–410
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

On computational proofs of the existence of solutions to nonlinear parabolic problems

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Received 9 June 1992; revised 2 March 1993

Abstract

This paper is an extension of the preceding study (Nakao, this journal, 1991) in which we described a numerical verification method of the solution for one-space dimensional parabolic problems, to the several-space dimensional case. Here, numerical verification means the automatic proof of the existence of solutions to the problems by some numerical techniques on a computer. We reformulate the verification condition for nonlinear parabolic initial boundary value problems using the fixed-point problem of a compact operator on certain function spaces. As in the preceding study based upon a simple C^0 finite-element approximation and its constructive a priori error estimates, a numerical verification procedure is presented with some numerical examples.

Key words: Parabolic problem; Finite-element method; Error estimates; Fixed-point theorem

1. Introduction

In recent years, several techniques have been developed to use computers in proving existence and/or uniqueness of exact solutions for functional equations. In particular, there are not a few approaches for ordinary differential equations and some of them have already attained a sufficiently practical level (e.g., [4]). For partial differential equations, such techniques have been studied in [6–8,11,12,14,15] for the elliptic case, as well as in [9,10] for evolution problems. The term *verification* implies that we can verify the exact solution near the approximate solution without any a priori assumptions on the existence of solutions of the original problem.

In the previous report [9], we proposed an approach to the numerical proof of existence of solutions for nonlinear parabolic initial boundary value problems of one space dimension.

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SSDI 0377-0427(93)E0135-9

However, there were some theoretical difficulties in applying the method to the several-space dimensional case. The present paper is a further improved version which overcomes such difficulties as well as contains some additional refinement.

First, setting appropriate function spaces, we reformulate the problem to be considered as the fixed point of a compact operator. Next, we introduce, as in [9], the concepts *rounding* and *rounding error*, which enable us to treat the infinite-dimensional problem by a finite procedure, i.e., by computer. They are defined by the C^0 finite-element approximation and the computable error estimates for some simple linear parabolic problem. Using these concepts, we give the computational verification conditions and describe the actual verification procedures in computer. Finally, some numerical examples for the two-space dimensional case are illustrated.

2. Problem and fixed-point formulation

Consider the following nonlinear parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x, t, u, \nabla u), & (x, t) \in \Omega \times J, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times J, \\ u(x, 0) = 0, & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a convex bounded domain in $\mathbb{R}^n$, $1 \leqslant n \leqslant 3$, with piecewise smooth boundary $\partial\Omega$ and $J = (0, T)$ with $T > 0$. Set $Q = \Omega \times J$. We denote the usual and the time-dependent Sobolev spaces by $H^s \equiv H^s(\Omega)$ and $H^r(J;\, H^s)$, respectively (see [3]). Furthermore, we set $H_0 \equiv L^2(J;\, H_0^1)$ and $H \equiv H^1(J;\, H_0^1) \cap L^\infty(Q)$, where H_0^1 denotes the subspace of H^1 with homogeneous boundary condition. The norm on H is defined by

$$\|u\|_H = \|u\|_{H^1(J;H_0^1)} + \|u\|_{L^\infty(Q)}.$$

Also define

$$\tilde{H} \equiv \left\{ u \in H \mid \lim_{t \to 0} u(t) = 0 \text{ in } L^\infty(\Omega) \text{ and } \lim_{t \to 0} \nabla u(t) = 0 \text{ in } L^2(\Omega) \right\}.$$

From now on, we use the abbreviations $\|\cdot\| \equiv \|\cdot\|_{L^2(Q)}$ and $\|\cdot\|_\Omega \equiv \|\cdot\|_{L^2(\Omega)}$.

We now suppose the following assumptions on f in (1).

(A1) For each bounded subset U of H, $f(\cdot, U, \nabla U) \equiv \{f(\cdot, u, \nabla u) \mid u \in U\}$ is also bounded in $H^1(J;\, L^2)$.

(A2) For each bounded subset U of H, f is a continuous map from U into $H^1(J;\, L^2)$ in the H-norm as well as into $L^2(J;\, L^2)$ in the H_0-norm.

(A3) $f(0) \equiv \lim_{t \to 0} f(\cdot, t, u, \nabla u)$, in $L^2(\Omega)$ sense, belongs to $H_0^1(\Omega)$ for any $u \in \tilde{H}$.

The typical example of f satisfying above assumptions is, when $n = 2$,

$$f(x, t, u, \nabla u) = p \cdot \nabla u + q u^m + \phi,$$

where p, q, ϕ are bounded and smooth functions on Q such that $\phi(\cdot, 0) \in H_0^1(\Omega)$.

Now for all $g \in L^2(\Omega \times J)$, define $\phi \equiv Ag \in H^1(J;\ L^2) \cap L^2(J;\ H^2 \cap H_0^1)$ by

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - \Delta\phi = g, & (x,\ t) \in \Omega \times J, \\ \phi(x,\ t) = 0, & (x,\ t) \in \partial\Omega \times J, \\ \phi(x,\ 0) = 0, & x \in \Omega. \end{cases} \tag{2}$$

Then we have the following property.

Lemma 1. *The map A defined above is continuous from $L^2(J;\ L^2)$ to $H^1(J;\ L^2) \cap L^2(J;\ H^2 \cap H_0^1)$ as well as from $H^1(J;\ L^2)$ to H. Furthermore, if $g \in H^1(J;\ L^2)$, then $Ag \in \tilde{H}$.*

Proof. In this proof, we use the symbol C to denote a generic positive constant not necessarily the same in any two places. The continuity from $L^2(J;\ L^2)$ to $H^1(J;\ L^2) \cap L^2(J;\ H^2 \cap H_0^1)$ is followed easily by a standard argument, e.g., [5]. We now show the continuity from $H^1(J;\ L^2)$ to H. Differentiating both sides of (2) in t, it is seen that ϕ_t is a solution of the following equation:

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = g_t, & (x,\ t) \in \Omega \times J, \\ v(x,\ t) = 0, & (x,\ t) \in \partial\Omega \times J, \\ v(x,\ 0) = g(\cdot,\ 0), & x \in \Omega. \end{cases} \tag{3}$$

Hence, we have (cf. [5]), for a positive constant $\hat{C}$,

$$\|\nabla\phi_t\| \leqslant C(\|g(\cdot,\ 0)\| + \|g_t\|) \leqslant \hat{C}\|g\|_{H^1(J;\ L^2)}. \tag{4}$$

Further, by the imbedding $H^2 \hookrightarrow L^\infty$ and the well-known estimates for the function in $H^2 \cap H_0^1$ (e.g., [2]), we have, for almost all $(x,\ t) \in \Omega \times J$,

$$\begin{aligned} |\phi(x,\ t)| &\leqslant C\|\phi(\cdot,\ t)\|_{H^2} \leqslant C\|\Delta\phi(\cdot,\ t)\|_{L^2} \\ &\leqslant C(\|\phi_t(\cdot,\ t)\| + \|g(\cdot,\ t)\|) \leqslant \tilde{C}\|g\|_{H^1(J;\ L^2)}, \end{aligned} \tag{5}$$

where $\tilde{C}$ is a certain positive constant. Here, we have used the a priori estimates for the solution of (2) (e.g., [5]) and the following identity:

$$g(\cdot,\ t) = g(\cdot,\ 0) + \int_0^t g_t(\cdot,\ s)\ \mathrm{d}s.$$

Next, the continuity of g in t implies that

$$\lim_{t \to 0} \|\phi(\cdot,\ t)\|_{L^\infty} \leqslant \lim_{t \to 0} C\|\Delta\phi(\cdot,\ t)\|_{L^2} = C\|\Delta\phi(\cdot,\ 0)\|_{L^2} = 0.$$

Also using the results in [5], particularly [5, Lemma 2.2], we have

$$\|\nabla\phi(\cdot,\ t)\|_{L^2}^2 \leqslant \|\nabla\phi(\cdot,\ 0)\|_{L^2}^2 + C\int_0^t \|g(\cdot,\ s)\|_{L^2}^2\ \mathrm{d}s.$$

Since the right-hand side of the above tends to 0 as $t \to 0$, we have $\phi \in \tilde{H}$. □

We now define the weak solution for (1) as $u \in \tilde{H}$ satisfying

$$(u_t, v)_\Omega + (\nabla u, \nabla v)_\Omega = (f(\cdot, u, \nabla u), v)_\Omega, \quad v \in H_0^1,\ t \in J. \tag{6}$$

Here, $(\cdot, \cdot)_\Omega$ implies the L^2 inner product on Ω. Then the weak solution of (1) can be rewritten in the fixed-point form: find $u \in \tilde{H}$ such that

$$u = Af(\cdot, u, \nabla u).$$

Then we obtain the following condition of existence of the solution for (1) corresponding to [9, Theorem 1].

Theorem 2. *If there exists a convex and nonempty subset $U \subset \tilde{H}$ which is bounded in H and satisfies*

$$Af(\cdot, U, \nabla U) \subset U,$$

then there exists a solution $u \in \overline{U}^0 \cap \tilde{H}$ to (1). *Here, $\overline{U}^0$ means the closure of U in the H_0-norm.*

Proof. Note that $\overline{U}^0$ is a bounded subset in H. Indeed, this can be shown as below.

First, set $M \equiv \sup_{\phi \in U} \|\phi\|_H$. For each fixed $u \in \overline{U}^0$, let $\{u_n\}$ be a sequence in U such that $u_n \to u$ in H_0 as $n \to \infty$. By the weak compactness of $U \subset H^1(J;\ H_0^1)$, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and an element $\hat{u} \in H^1(J;\ H_0^1)$ such that $u_{n_j} \rightharpoonup \hat{u}$ (weakly) in $H^1(J;\ H_0^1)$ as $j \to \infty$. It also implies [18] that, from the compactness of the imbedding $H^1(J;\ H_0^1) \hookrightarrow L^2(J;\ L^2)$, $u_{n_j} \to \hat{u}$ in $L^2(J;\ L^2)$ as $j \to \infty$. On the other hand, since u_n converges to u in $L^2(J;\ L^2)$, we have $u = \hat{u}$.

Also, the property of weak limit yields that

$$\|u\|_{H^1(J;H_0^1)} \leqslant \lim_{j \to \infty} \inf \|u_{n_j}\|_{H^1(J;H_0^1)} \leqslant M.$$

Furthermore, $\|u\|_{L^\infty(Q)} \leqslant M$ follows from the fact that there exists a subsequence of $\{u_n\}$, bounded by M, which converges to u pointwise almost everywhere in Q. Therefore, from the assumption (A2) and Lemma 1, it is seen that the composite map $Af : \overline{U}^0 \to H^1(J;\ L^2) \cap L^2(J;\ H^2 \cap H_0^1)$ is continuous in the H_0-norm. Thus we have

$$Af(\cdot, \overline{U}^0, \nabla \overline{U}^0) \equiv Af(\overline{U}^0) \subset \overline{Af(U)}^0 \subset \overline{U}^0.$$

Since, using the compactness of the imbedding $H^1(J;\ L^2) \cap L^2(J;\ H^2) \hookrightarrow H_0$, Af is a compact map on the bounded, convex and closed subset $\overline{U}^0$ in H_0, by Schauder's fixed-point theorem, there exists an element $u \in \overline{U}^0$ such that $u = Af(\cdot, u, \nabla u)$. Moreover, noting that $\overline{U}^0 \subset H$, from the assumption (A1) and Lemma 1, we have $Af(\cdot, u, \nabla u) \in \tilde{H}$ which implies $u \in \overline{U}^0 \cap \tilde{H}$. □

3. Rounding and verification conditions

Most of this section is similar to the description in the corresponding part of [9].

First, for parameter h, $0 < h < 1$, let $S_{x,h} \subset H_0^1(\Omega)$ and $S_{t,h} \subset \tilde{H}^1(J) \equiv \{v \in H^1(J) \mid v(0) = 0\}$ be the piecewise linear finite-element subspaces on Ω and J, respectively, satisfying

$$\inf_{\chi \in S_{x,h}} \|u - \chi\|_{H_0^1} \leqslant C_1 h\, | u |_{H^2(\Omega)}, \quad \forall u \in H^2 \cap H_0^1, \tag{7}$$

and

$$\inf_{\eta \in S_{t,h}} \|v-\eta\|_{L^2} \leqslant \begin{cases} C_2 h^2 \|\ddot{v}\|_{L^2(J)}, & \forall v \in H^2(J) \cap \tilde{H}^1(J), \\ C_3 h \|\dot{v}\|_{L^2(J)}, & \forall v \in \tilde{H}^1(J), \end{cases} \tag{8}$$

where $|u|_{H^2(\Omega)} = \sum_{i,j=1}^{n} \|\partial^2 u/(\partial x_i \partial x_j)\|_{\Omega}^2$. Here, C_1–C_3 are supposed to be positive constants which can be numerically determined independent of h.

We now adopt $S_h \equiv S_{x,h} \otimes S_{t,h}$ as the approximation space on $\Omega \times J$. Then, for $g \in L^2(J;\ L^2)$, the rounding $u^h \equiv R(Ag) \in S_h$ as

$$\int_0^T \{(u_t^h, v)_\Omega + (\nabla u^h, \nabla v)_\Omega\}\, \mathrm{d}t = \int_0^T (g, v)_\Omega\, \mathrm{d}t, \quad v \in S_h. \tag{9}$$

It is easily seen that there exists a unique u^h satisfying (9).

Then we have the following explicit error estimates corresponding to the results in [9].

Lemma 3. *For $g \in H^1(J;\ L^2)$ with $g(\cdot, 0) \in H_0^1$, let u and u^h be solutions of* (2) *and* (9), *respectively. Then, there exists a positive constant C such that*

$$\|\nabla(u-u^h)\| \leqslant Ch, \tag{10}$$

where $C = C(g, \|u_t^h\|)$ is given by

$$C^2 = 2\left\{K_1\left(C_1^2 K_2 + C_2\sqrt{K_3}\right) + \tfrac{1}{2}\left(C_1 K_2 + C_3\sqrt{K_4}\right)^2\right\}, \tag{11}$$

where

$$\begin{cases} K_1 = \|g\| + \|u_t^h\|, \\ K_2 = 4\|g\|, \\ K_3 = \|\nabla g(\cdot, 0)\|_\Omega^2 + \|g_t\|^2, \\ K_4 = \tfrac{1}{2}\left(\|g(\cdot, 0)\|_\Omega^2 + \|g\|^2 + \|g_t\|^2\right). \end{cases} \tag{12}$$

The proof is quite analogous to that of [9, Lemma 2].

Thus, based upon the error estimates in Lemma 3, we define the rounding error RE(Ag) as

$$\mathrm{RE}(Ag) = \left\{\phi \in \tilde{H} \mid \|\phi\|_{H_0} \leqslant Ch\right\}, \tag{13}$$

where C is the same constant as in Lemma 3. Moreover, $R(AG)$ and RE(AG) for the set of functions $G \subset H^1(J;\ L^2)$ with $G(\cdot, 0) \subset H_0^1$ are defined as $R(AG) = \{R(Ag) \mid g \in G\}$ and $\mathrm{RE}(AG) = \bigcup_{g \in G} \mathrm{RE}(Ag)$, respectively.

Now let $\{\phi_j\}_{j=1,\dots,M}$ be a basis of S_h and let $\mathscr{S}_{h,I}$ denote the set of all linear combinations of $\{\phi_j\}$ with interval coefficients. That is, $\phi_h = \sum_{j=1}^{M} [\underline{A}_j, \overline{A}_j]\phi_j \in \mathscr{S}_{h,I}$ means that

$$\phi_h = \left\{\hat{\phi}_h \in S_h \mid \hat{\phi}_h = \sum_{j=1}^{M} a_j \phi_j,\ a_j \in \left[\underline{A}_j, \overline{A}_j\right],\ 1 \leqslant j \leqslant M\right\}.$$

And for $\alpha \in \mathbb{R}^+$, set $[\alpha] \equiv \{\phi \in \tilde{H} \mid \|\phi\|_{H_0} \leqslant \alpha\}$. Further, for $U_h \in \mathscr{S}_{h,I}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$, we define

$$D(U_h, \alpha)_{\beta,\gamma,\delta} \equiv \left\{\phi \in \tilde{H} \mid \phi \in U_h + [\alpha] \text{ with } \|\phi_t\| \leqslant \beta,\ \|\nabla\phi_t\| \leqslant \gamma,\ \|\phi\|_{L^\infty(Q)} \leqslant \delta\right\}.$$

Then we have the following verification condition.

Theorem 4. *Let $U \equiv D(U_h, \alpha)_{\beta,\gamma,\delta}$ be a subset in $\tilde{H}$ for some $U_h \in \mathscr{S}_{h,I}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ defined above, and set $G \equiv f(\cdot, U, \nabla U)$. If the following is valid, then there exists a solution $u \in U$ to* (1):

$$R(AG) \subset U_h, \tag{14}$$

$$\|\mathrm{RE}(AG)\|_{H_0} \leqslant \alpha, \tag{15}$$

$$\|G\| \leqslant \beta, \tag{16}$$

$$\hat{C}\|G\|_{H^1(J;\, L^2)} \leqslant \gamma, \tag{17}$$

$$\tilde{C}\|G\|_{H^1(J;\, L^2)} \leqslant \delta. \tag{18}$$

Here, the constants $\hat{C}$ and $\tilde{C}$ are the same as defined by (4) and (5) in the proof of Lemma 1, respectively. And the norm for the set of functions implies the supremum value for all elements.

Proof. Note that, for any $g \in G$, by the conditions (14), (15) and the definitions of $R(Ag)$ and $\mathrm{RE}(Ag)$, we have

$$R(Ag) + \mathrm{RE}(Ag) \in U_h + [\alpha].$$

Also taking notice of the a priori estimates in [5] and the estimates (4) and (5), assumptions (16)–(18) yield that

$$\|(Ag)_t\| \leqslant \beta, \qquad \|\nabla(Ag)_t\| \leqslant \gamma, \qquad \|Ag\|_{L^\infty(Q)} \leqslant \delta,$$

respectively. Therefore, $Af(\cdot, U, \nabla U) \subset U$ holds. Thus by Theorem 2, there exists a fixed point u of Af in $\overline{U}^0 \cap \tilde{H}$. Finally, from the proof of Theorem 2, we have $u \in D(U_h, \alpha)_{\beta,\gamma,\delta}$. □

4. Verification procedure by computer

In this section, we describe a concrete algorithm for the generation of the set which satisfies the verification conditions (14)–(18). We use an iterative procedure similar to that in [9].

First, $u_0^h \in S_h$ and $\alpha_0 \in \mathbb{R}^+$ are appropriately taken; normally, u_0^h is chosen as a finite-element solution of (1) and as $\alpha_0 = 0$. Also we set $\beta_0 = \|(u_0^h)_t\|$, $\gamma_0 = \|(\nabla u_0^h)_t\|$, $\delta_0 = \|u_0^h\|_{L^\infty(Q)}$ and $U_0 = D(\{u_0^h\}, \alpha_0)_{\beta_0,\gamma_0,\delta_0}$.

Next, let ϵ be a small and fixed positive number. When $i \geqslant 1$, for $u_{i-1}^h = \sum_{j=1}^M [\underline{A}_j^{(i-1)}, \overline{A}_j^{(i-1)}]\phi_j$, set

$$\tilde{u}_{i-1}^h \equiv \sum_{j=1}^{M} \left[\underline{A}_j^{(i-1)} - \epsilon,\ \overline{A}_j^{(i-1)} + \epsilon\right]\phi_j, \tag{19}$$

$$\tilde{\alpha}_{i-1} \equiv \alpha_{i-1} + \epsilon, \tag{20}$$

$$\tilde{\beta}_{i-1} \equiv \beta_{i-1} + \epsilon, \tag{21}$$

$$\tilde{\gamma}_{i-1} \equiv \gamma_{i-1} + \epsilon, \tag{22}$$

$$\tilde{\delta}_{i-1} \equiv \delta_{i-1} + \frac{\tilde{C}}{\hat{C}}\epsilon. \tag{23}$$

And set $\tilde{U}_{i-1} = D(\tilde{u}_{i-1}^h, \tilde{\alpha}_{i-1})_{\tilde{\beta}_{i-1},\tilde{\gamma}_{i-1},\tilde{\delta}_{i-1}}$, i.e., ϵ-inflation of U_{i-1} (cf. [16]). We now determine the ith iteration $U_i = D(u_i^h, \alpha_i)_{\beta_i,\gamma_i,\delta_i}$ by choosing $u_i^h \in \mathscr{S}_{h,I}$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}^+$ as

$$\int_0^T \left\{ \left((u_i^h)_t, \phi_j \right)_\Omega + \left(\nabla u_i^h, \nabla \phi_j \right)_\Omega \right\} \mathrm{d}t \supset \int_0^T \left(\tilde{G}_{i-1}, \phi_j \right)_\Omega \mathrm{d}t, \quad 1 \leqslant j \leqslant M, \tag{24}$$

$$\alpha_i = hC\left(\tilde{G}_{i-1}, \|(AG_{i-1})_t\|\right), \tag{25}$$

$$\beta_i = \|\tilde{G}_{i-1}\|, \tag{26}$$

$$\gamma_i = \hat{C}\|\tilde{G}_{i-1}\|_{H^1(J;\, L^2)}, \tag{27}$$

$$\delta_i = \frac{\tilde{C}}{\hat{C}}\gamma_i, \tag{28}$$

where $\tilde{G}_{i-1} \equiv f(\cdot, \tilde{U}_{i-1}, \nabla\tilde{U}_{i-1})$, and C, $\hat{C}$ and $\tilde{C}$ are the constants appearing in (10), (4) and (5), respectively. Here, the right-hand side of (24) is evaluated as an interval in the over-estimated sense as in [9] (see also [6–8], etc.). Then (24) is reduced to solving a system of linear equations with interval right-hand sides. And thus u_i^h is determined by an interval vector solution of such an equation using the usual interval approaches (e.g., [1,13,16]).

Now, from Theorem 4, we have the following condition for the completion of the verification.

Theorem 5. *If, for some integer N,*

$$u_N^h \subset \tilde{u}_{N-1}^h, \qquad \alpha_N \leqslant \tilde{\alpha}_{N-1}, \qquad \beta_N \leqslant \tilde{\beta}_{N-1}, \qquad \gamma_N \leqslant \tilde{\gamma}_{N-1},$$

then there exists a solution u of (1) in U_N, where $U_N = D(u_N^h, \alpha_N)_{\beta_N,\gamma_N,\delta_N}$.

Remark 6. Note that the inequality $\delta_N \leqslant \tilde{\delta}_{N-1}$ necessarily holds from the condition in the above theorem. Indeed, this is easily checked by some simple calculations using the definition of the ϵ-inflation for δ_i, i.e., (23).

5. Verification examples

We consider the following nonlinear equation with two space variables in a rectangular domain:

$$\begin{cases} u_t - \Delta u = p \cdot \nabla u + \lambda u^2 + r \sin t, & (x, t) \in \Omega \times J, \\ u(x, 0) = 0, & x \in \Omega, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times J, \end{cases} \tag{29}$$

where $\Omega = (0, 1) \times (0, 1)$ and $J = (0, 1)$. Also, $p = (p_1, p_2)$, λ and r are given constants.

Let δ_x: $0 = x_0 < x_1 < \cdots < x_L = 1$ be a uniform partition of the interval (0, 1) in the x-direction and, for simplicity, we set $\delta_y = \delta_x$ and $\delta_t = \delta_x$. Then $h = 1/L$. Let $\mathscr{M}_1(x)$ denote the set of continuous piecewise linear functions on (0, 1) in the x-direction and set $S_x = \{v \in \mathscr{M}_1(x) \mid v(0) = v(1) = 0\}$, S_y the same in the y-direction. Also set $S_{x,h} = S_x \otimes S_y$. Thus, setting $S_{t,h} = \{v \in \mathscr{M}_1(t) \mid v(0) = 0\}$, define $S_h \equiv S_{x,h} \otimes S_{t,h}$.

Then, by using well-known results (e.g., [6,17,19]), the values of constants C_1–C_3 in Section 3 can be taken as

$$C_1 = \frac{1}{\pi}, \qquad C_2 = \frac{1}{\pi^2}, \qquad C_3 = \frac{1}{\pi}.$$

We also adopt the usual hat functions as the basis of $\mathscr{M}_1(k)$, where $k = x$, y and t.

Now we add some remarks to the actual computation of the right-hand sides of (24)–(28) (cf. [9]). For example, for an arbitrary $\alpha \in [\tilde{\alpha}_{i-1}]$, $\|\alpha^2\|$ can be estimated as below.

First, using the imbedding constant [7] for $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we have, for any $1 < p < \infty$,

$$\|\alpha\|_{L^p(\Omega)} \leqslant \frac{1}{\sqrt{2}} \|\nabla\alpha\|_\Omega. \tag{30}$$

Next, notice that α can be considered as $u - u^h$, where $u = Ag$ and $u^h = R(Ag)$ for some $g \in \tilde{G}_{i-2}$. Hence, (30) implies

$$\begin{aligned}\|\alpha^2\|^2 &\leqslant \int_J \tfrac{1}{4}\|\nabla\alpha\|_\Omega^4 \,\mathrm{d}t \leqslant \tfrac{1}{4}\left(\int_J \|\nabla\alpha\|_\Omega^2 \,\mathrm{d}t\right)\|\nabla\alpha\|^2_{L^\infty(J;L^2)} \\ &\leqslant \tfrac{1}{4}\alpha^2\left(\|\nabla u^h\|_{L^\infty(J;L^2)} + \|\nabla u\|_{L^\infty(J;L^2)}\right)^2 \leqslant \tfrac{1}{4}\tilde{\alpha}_{i-1}^2\left(\|\nabla \tilde{u}_{i-1}^h\|_{L^\infty(J;L^2)} + \tilde{\beta}_{i-1}\right)^2,\end{aligned}$$

which yields the estimation of $\|\alpha^2\|$.

Also, we can estimate the constant $\hat{C}$ in (4) as below. For $u = Ag$, by differentiating the following equation in t and setting $v = u_t$,

$$(u_t, v)_\Omega + (\nabla u, \nabla v)_\Omega = (g, v)_\Omega, \quad v \in H_0^1(\Omega),\ t \in J,$$

we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|_\Omega^2 + \|\nabla u_t\|_\Omega^2 \leqslant \tfrac{1}{2}\left(\|g_t\|_\Omega^2 + \|g\|_\Omega^2\right). \tag{31}$$

Here, we have used $\|u_t\|_\Omega^2 \leqslant \|g\|_\Omega^2$. Integrating the above from 0 to T and using $\|u_t(0)\|_\Omega = \lim_{t\to 0}\|u_t(t)\|_\Omega = \lim_{t \to 0}\|g(\cdot, t)\|_\Omega = 0$, one obtains

$$\|\nabla u_t\|^2 \leqslant \tfrac{1}{2}\|g\|^2_{H^1(J;L^2)},$$

which implies $\hat{C} \leqslant 1/\sqrt{2}$.

We now determine the constant $\tilde{C}$ in (5). Observe that by using (31),

$$\begin{aligned}u(x, y, t) &= \int_0^x \int_0^y u_{xy}(\xi, \eta, t)\,\mathrm{d}\eta\,\mathrm{d}\xi \leqslant \|\Delta u(\cdot, t)\|_\Omega \\ &\leqslant \|u_t(\cdot, t)\|_\Omega + \|g(\cdot, t)\|_\Omega \leqslant \sqrt{\|g_t\|^2 + \|g\|^2} + \|g_t\|.\end{aligned}$$

Here, we have used [2] the estimates $\|u_{xy}(\cdot, t)\|_\Omega \leqslant \|\Delta u(\cdot, t)\|_\Omega$. Therefore, we get $\tilde{C} \leqslant 2$.

Furthermore, in order to obtain the initial approximation u_0^h, we used a kind of simplified spectral method. That is, for given parameters p, λ, r, set

$$u(x, y, t) = C \sin(\pi x) \sin(\pi y) \sin(\tfrac{1}{2}\pi t)$$

and substitute it into the function of the form $f(u) \equiv u_t - \Delta u - p \cdot \nabla_u - \lambda u^2 - r \sin t$. Then we solve the equation

$$\int_0^1 (f(u), u)_\Omega \, dt = 0,$$

which is equivalent to a quadratic equation for C, once parameters p, λ and r are given. This fact suggests that problem (29) has two branch solutions. u_0^h is obtained by the usual interpolation of the above $u(x, y, t)$ on each node. We could actually verify the solutions for several cases. We now illustrate some of these numerical results.

Case 1: Data: $p = (0.01, 0.01)$, $\lambda = 0.1$, $r = 2.5$.

Result: number of partitions = 12 (dim $S_h = 1452$); verified with $N = 38$ iterations; maximum absolute value of coefficient intervals in $u_N^h = 1.04$; H_0-error bound $(\alpha_N) = 0.32$; L^2 bound of u_t $(\beta_N) = 1.41$; L^2 bound of ∇u_t $(\gamma_N) = 3.874$; L^∞ bound of u $(\delta_N) = 10.9$.

Case 2: Data: $p = (0.01, 0.01)$, $\lambda = 0.3$, $r = 1.0$.

Result: number of partitions = 8 (dim $S_h = 392$); verified with $N = 31$ iterations; maximum absolute value of coefficient intervals in $u_N^h = 0.20$; H_0-error bound $(\alpha_N) = 0.19$; L^2 bound of u_t $(\beta_N) = 0.56$; L^2 bound of ∇u_t $(\gamma_N) = 1.99$; L^∞ bound of u $(\delta_N) = 5.62$.

Remark 7. The above results are considered as the verification of lower-branch solutions. To verify the upper-branch solution, we will need some Newton type method as in [8,11]. The H_0-error bound seems not to be so good in these examples, because we could only use rather rough meshes by the limitation of our computer facility. But, as it is seen from the arguments in previous sections, the accuracy is clearly $O(h)$. By this fact and the use of the residual technique (e.g., [19]), we will be able to obtain more accurate error bounds for finer meshes. On the other hand, since the present method essentially has no consideration for the bounds of $\| u_t \|$, $\| \nabla u_t \|$ and $\| u \|_{L^\infty}$, we need some other techniques to refine these bounds.

Remark 8. In these calculations, we used computer arithmetic with usual double precision instead of strict interval computations (e.g., ACRITH [20], PASCAL-SC, etc.). That is, we neglected the round-off error of the floating-point arithmetic in the verification procedure (24)–(28). But from our experiences, the order of magnitude for the effect of round-off error is under 10^{-10}. Therefore, it is almost negligible compared with the truncation error which amounts to around 10^{-1} in the present case. Of course, we have to devise some rigourous numerical computations, which means the computation with guaranteed accuracy including the round-off error, when we apply this method for the mathematical proof of the real problem.

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